Derived Schemes.

Before discussing deformation theory we will introduce a class of prestacks that are "natural" generalizations of classical schemes. The philosophy here is that derived schemes are prestacks w/ certain properties, yet we don't need to introduce any extra structures, for instance what a locally ringed ∞-topos is.

<u>Def'n:</u> A prestack Z is said to be a <u>scheme</u> if

(i) Z satisfies étale descent;

(ii) $Z \to Z \times Z$ is affine schematic, and $\forall\, T \xrightarrow{\text{aff.}} Z \times Z$ the map
$$^{cl}\left(T \underset{Z \times Z}{\times} Z\right) \to {}^{cl}T$$
is a closed embedding;

(iii) $\exists$ a collection $\{f_i : S_i \to Z\}_I$ (Zariski atlas) s.t.

- f_i is an open embedding,
- $\coprod_I {}^{cl}(T \underset{Z}{\times} S_i) \to {}^{cl}T$ cover ${}^{cl}T$ $\forall\, T \xrightarrow{\text{aff.}} Z$.

We let $Sch \hookrightarrow Stk \hookrightarrow PStk$ denote the subcategory of schemes.

<u>Rk:</u> 1. (i)' Z satisfy Zariski descent $\Rightarrow$ (i) Z satisfy étale desc. $\Rightarrow$ (i)'' Z satisfy flat desc.

2. We are restricting to the case of separated in this definition for ease of exposition. The non separated cases will be included in n-geometric stacks to be defined later.

We discuss some formal properties of the definition.

For Z a scheme and $\{f_i : S_i \to Z\}$ a Zariski atlas, one has that

$$\coprod_I S_i \to Z \quad \text{is an étale surjection.}$$

$$\Rightarrow \quad L\Big(\big|\big(\coprod_I S_i\big)^\bullet / Z\big|_{pStk}\Big) \xrightarrow{\simeq} Z. \ (\bigstar).$$

Lemma: (Lifting open embeddings / étale morphisms.) Let $f : Z' \to Z$ be an affine Zariski or affine étale morphism between schemes. For $T \to Z$, where $T \in Sch^{aff}$, one has an equivalence (of spaces).

$$\left\{\begin{tikzcd} & Z' \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}\right\} \simeq \left\{\begin{tikzcd} & {}^{cl}Z' \arrow[d] \\ {}^{cl}T \arrow[r] \arrow[ur, dashed] & {}^{cl}Z \end{tikzcd}\right\}.$$

Pf: Indeed, for $Z' \to Z$ in Sch^{aff} this follows from the equivalence of the (small) étale site over S & ${}^{cl}S$. For schemes this follows from using the presentation $(\bigstar)$.

The presentation of $(\bigstar)$ is useful to determine when one has a scheme for a given underlying classical scheme.

Digression: given an ∞-category $\mathcal{C}$ a groupoid object in $\mathcal{C}$ is a functor

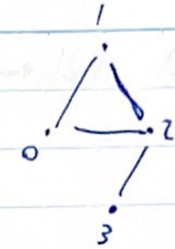
$X : \Delta^{op} \to \mathcal{C}$ s.t. $\forall n \geq 0$ & a partition $[n] = S \cup S'$ w/ $S \cap S' = \{s\}$ one has.

$$X([n]) \xrightarrow{\simeq} X(S) \times_{X(\{s\})} X(S') \quad \text{in } \mathcal{C}.$$

E.g.: $- \ X([0,2]) \xrightarrow{\simeq} X([1]) \times_{t, X([0]), s} X([1]).$

$- X([3]) \xrightarrow{\simeq} X([2]) \times_{X([0])} X([1])$

$\Rightarrow X([3]) \to X([1]) \times_{X([0])} X([1]) \times_{X([0])} X([1])$.



FACT: In $\mathcal{C} = S_{qc}$, or more generally $\mathcal{C}$ an ∞-topos, e.g. $\mathcal{C} = PStk$ or Stk, any groupoid object $X_\bullet : \Delta^{op} \to \mathcal{C}$ is effective, i.e.

$$X \simeq (X_0/X_{-1})^\bullet \quad \text{where} \quad X_{-1} := \operatorname*{colim}_{[n]\in\Delta^{op}} X_n \quad \text{and}.$$

$(X_0/X_{-1})^\bullet$ is the Čech nerve, i.e. $\left(\cdots \Rrightarrow X_0 \times_{X_{-1}} X_0 \rightrightarrows X_0\right)$.

in a topos

This result should be compared w/ the fact in ordinary category thy. that any equivalence relation is effective, i.e. $R \subseteq X \times X$ an equivalence relation then $p: R \to X \times X$ is s.t. $R \simeq X \times_{X/R} X$.

$\forall S \in C$ $\quad Hom(S,R) \to Hom(S,X) \times Hom(S,X)$ is an eq. rel.

Let $S^\bullet : \Delta^{op} \to PStk$ be a groupoid object in $PStk$.

Consider $Z := L(|S^\bullet|)$.

Prop: • If $S^0 = \coprod_I S^0_i$, $S^1 = \coprod_J S^1_j$ are disjoint unions of affine schemes & each $S^1_j \to S^0_i$ is an open embedding.

• ${}^{cl}Z$ is a scheme, & $\coprod_I {}^{cl}S_i \to {}^{cl}Z$ is a Zariski atlas.

Then: (i) Z is a scheme;

(ii) $\coprod_J S_i^0 \to Z$ is a Zariski atlas.

Idea: (a) It is enough to show that each $S_i^0 \to Z$ is an affine open embedding. (Exercise.)

(b) $\mathcal{X} \to \mathcal{Y}$ in $PStk$ is affine schematic $\Rightarrow L(\mathcal{X}) \to L(\mathcal{Y})$ is affine schematic. (Consequence of the fact that affine schemes are sheaves.)

(c) Enough to check $S_i^0 \to |S^\bullet|$ is an aff. open embedding.

(d) Notice for $T \in Sch^{aff}$, $T \to |S^\bullet|$ factors through S^0, since
$T \to |S^\bullet| \simeq \operatorname{colim}_{\Delta^{op}} |S^\bullet(T)|$.

So $T \times_{|S^\bullet|} S^i \simeq T \times_{S^0} S^0 \times_{|S^\bullet|} S^i$.

Thus, it is enough to prove $S^0 \times_{|S^\bullet|} S^i \to S^0$ is an affine open embedding.

But $S^0 \times_{|S^\bullet|} S^i \simeq \left(S^0 \times_{|S^\bullet|} S^0\right) \times_{S^0} S^i$ & $S^0 \times_{|S^\bullet|} S^0 \simeq S^1$.

And the result follows for $S^1 \to S^0$ being affine open. ■

<u>Example:</u> Let $R = k[y]$, $|y| = -2$. $S = \operatorname{Spec} R$, ${}^{cl}S = \operatorname{Spec} k$.

Consider: $S_\bullet^0 := \operatorname{Spec} k[y][x] \amalg \operatorname{Spec} k[y][Y]$.

$x \mapsto x$, $Y \mapsto x^{-1}$

$S_\bullet^1 := \operatorname{Spec} k[y][x, x^{-1}]$.

$S_\bullet^2 := S_\bullet^1 \times_{S_\bullet^0} S_\bullet^1$, $S_\bullet^3 = (\dots)$.

Then, $L(|S^\bullet|) \simeq \mathbb{P}^1_R$ is a derived scheme.

<u>Corollary:</u> For $Z \in Sch$.

(i) $\{f: Z' \to Z \mid f \text{ affine Zariski}\} \simeq \{f_0 : Z'_0 \to {}^{cl}Z \mid f_0 \text{ affine Zariski}\}$.

(ii) Z' is affine $\iff$ ${}^{cl}Z'$ is affine.

(iii) $Z \in Sch^{aff} \iff {}^{cl}Z \in {}^{cl}Sch^{aff}$.

Now we discuss how the scheme condition interacts w/ ~~the~~ ~~eoc~~ coconnective & truncatedness condition. We start w/ the following sanity check:

<u>Lemma:</u> Let $Z_0 \in {}^{cl}PStk := Fun(({}^{cl}Sch^{aff})^{op}, Spc)$ s.t.

(i) Z_0 satisfy étale descent;

(ii) $Z_0 \to Z_0 \times Z_0$ is aff. schematic & $\forall \overset{\text{classical aff.}}{T_0} \to Z^0$, $T_0 \times_{Z^0 \times Z^0} Z^0 \overset{\text{closed}}{\hookrightarrow} T_0$,

(iii) $\exists \coprod_I S_i \to Z_0$ a Zariski cover.

We claim Z_0 is a classical scheme. In particular, Z_0 is 0-truncated.

Proof: ~~Let $T_0 \in {}^{cl}Sch^{aff}$ we claim $Z_0(T_0)$ is k-truncated $\Leftrightarrow$ $T_0 \times_{Z_0 \times Z_0} Z_0$ is $(k-1)$-truncated~~

Notice that $Z \simeq L(|S^\cdot|_{PStk})$ is 0-truncated if $|S^\cdot|_{PStk}$ is 0-truncated.
for $T_0 \in {}^{cl}Sch^{aff}$

$|S^\cdot|_{PStk}(T_0) = |S^\cdot(T_0)|_{Spc}$. $S^\cdot(T_0)$ satisfies the Kan condition, i.e. $S^\cdot(T_0)(\Delta^n) \to S^\cdot(T_0)(\Lambda^n_i)$ induces a surjection on π_0.

(It is actually an isom. since $S^\cdot(T_0)$ is a groupoid object in Spc.)

Thus,

$$\pi_n |S^\cdot(T_0)|_{Spc} = \pi_0(Fib(S^\cdot(T_0)(\Delta^n) \to S^\cdot(T_0)(\partial\Delta^n))).$$

In particular, one has $\pi_1 |S^\cdot(T_0)|_{Spc} = \pi_0(Fib(Hom_{{}^{cl}Sch^{aff}}(T_0, S^1) \xrightarrow{\alpha} Hom_{{}^{cl}Sch^{aff}}(T_0, S^0) \times Hom_{{}^{cl}Sch^{aff}}(T_0, S^0)))$.

but α is injective since $S^1 \simeq S^0 \times S^0$ & $Z \to Z \times Z$ affine closed imply that $S^1 \to S^0 \times S^0$ is $\overset{Z}{\text{locally}}$ closed.

Similarly, $\pi_k(|S^\cdot(T_0)|_{Spc}) = 0 \quad \forall k \geq 2$. ■

for $Z \in {}^{\leq n}PStk \cap Sch =: {}^{\leq n}Sch$

~~More generally~~ Furthermore, one can prove that given $f_0 \in Z(T_0)$

$$\left\{ \begin{array}{ccc} T & \overset{f}{\dashrightarrow} & \\ \uparrow & & \\ T_0 & \xrightarrow{f_0} & Z \end{array} \right\}$$

the space of lifts f, where $T \in {}^{\leq n}Sch^{aff}$ is n-truncated.

This gives $Z \in {}^{\leq n}Sch$ is n-truncated. ■

As with stacks one says a scheme $Z \in Sch$ is n-coconnective, if it is n-coconnective as a stack, i.e.

$$^{L}LKE_{{}^{\leq n}Sch^{aff} \hookrightarrow Sch^{aff}}(Z|_{{}^{\leq n}Sch^{aff}}) \xrightarrow{\sim} Z.$$

We summarize some results related to this:

Cor: $Z \in Sch$ is n-coconnective iff one of $\exists\, S^0 \to Z$ a Zariski cover w/ $S^0 \simeq \coprod_I S_i^0$, w/ $S_i^0 \in {}^{\leq n}Sch^{aff}$.

$LKE_{{}^{\leq n}Sch^{aff} \hookrightarrow Sch^{aff}}$ identifies ${}^{\leq n}Sch$ w/ n-coconnective schemes.

We also mention the relation between the above condition & convergence. [See GR-I, Chapter 2 §3.4].

Prop: For $Z \in Sch$, Z is convergent.

Also given Z a convergent prestack, if $\forall n \geq 0$ ${}^{\leq n}Z \in {}^{\leq n}Sch$, then $Z \in Sch$.

We omit the discussion of finiteness conditions for the moment and just define:

a morphism $f: X \to Y$ of prestacks is schematic if $\forall S \in Sch^{aff}$ & $S \to Y$ the pullback $X \times_Y S \in Sch$.

Clearly, one can prove:

Lemma: For $X \to Z$ a schematic map of prestacks, if $Z \in Sch$ then $X \in Sch$.